

# VIRTUALLY COMPACT SPECIAL HYPERBOLIC GROUPS ARE CONJUGACY SEPARABLE

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**ABSTRACT.** We prove that any word hyperbolic group which is virtually compact special (in the sense of Haglund and Wise) is conjugacy separable. As a consequence we deduce that all word hyperbolic Coxeter groups and many classical small cancellation groups are conjugacy separable.

To get the main result we establish a new criterion for showing that elements of prime order are conjugacy distinguished. This criterion is of independent interest; its proof is based on a combination of discrete and profinite (co)homology theories.

## 1. INTRODUCTION

One of the main themes of Geometric Group Theory is the study of groups which act on non-positively curved spaces. Two prominent classes of such groups is the class of hyperbolic groups (defined by Gromov in [13]) and the class of (virtually) special groups (introduced by Haglund and Wise in [16]). The intersection of these two classes is quite large and its elements, virtually special hyperbolic groups, have particularly nice properties.

Recall that a finitely generated group  $G$  is said to be *hyperbolic* if its Cayley graph is a  $\delta$ -hyperbolic metric space, for some  $\delta \geq 0$  (see, for example, [2]). On the other hand,  $G$  is *virtually compact special*, if there is a finite index subgroup  $H \leq G$ , such that  $H$  is isomorphic to the fundamental group of a compact *special cube complex*, whose hyperplanes satisfy certain combinatorial properties (see [16, Sec. 3]).

Since the original work of Haglund and Wise [16], many hyperbolic groups have been shown to be virtually special. For example, in the paper [15] Haglund and Wise showed that hyperbolic Coxeter groups are virtually compact special. In [34] Wise proved the same for finitely generated 1-relator groups with torsion, while in [1] Agol showed this for fundamental groups of closed hyperbolic 3-manifolds. In fact, Agol [1] proved that any hyperbolic group admitting a proper cocompact action on a CAT(0) cube complex is virtually compact special.

In this paper we study conjugacy separability of virtually compact special hyperbolic groups. Recall, that a group  $G$  is *conjugacy separable* if for arbitrary non-conjugate elements  $x, y \in G$  there is a homomorphism from  $G$  to a finite group  $F$  such that the images of  $x$  and  $y$  are not conjugate in  $F$ . Conjugacy separability can be regarded

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as an algebraic analogue of solvability of the conjugacy problem in a group and has a number of applications. Most prominently it is used in proving residual finiteness of outer automorphism groups (see, for example, the discussion in [25, Sec. 2]).

Conjugacy separability is usually not easy to show, and, until recently, only a few classes of groups were known to satisfy it: virtually free groups [10], virtually surface groups [23] and virtually polycyclic groups [11, 29]. Note that in general conjugacy separability does not pass to finite index overgroups [12] or to finite index subgroups [24], therefore the adjective “virtually” is important.

A group  $G$  is said to be *hereditarily conjugacy separable* if every finite index subgroup of  $G$  is conjugacy separable. In [25] the first author showed that right angled Artin groups are hereditarily conjugacy separable. This result was subsequently used to prove conjugacy separability of Bianchi groups [7], 1-relator groups with torsion [26] and fundamental groups of compact 3-manifolds [17]. In fact, in [25] it was shown that any virtually compact special group  $G$  contains a conjugacy separable subgroup of finite index. But it is still unclear whether such  $G$  must necessarily be conjugacy separable itself. In the present paper we prove this in the case when  $G$  is hyperbolic:

**Theorem 1.1.** *Any virtually compact special hyperbolic group is hereditarily conjugacy separable.*

Conjugacy separability of torsion-free virtually compact special hyperbolic groups was proved in [25, Cor. 9.11], so the actual novelty of Theorem 1.1 is in handling groups with torsion. In view of Agol’s result [1, Thm. 1.1], the above theorem shows that every hyperbolic group, admitting a proper cocompact action on a CAT(0) cube complex, is hereditarily conjugacy separable. This gives an abundance of new examples of (hereditarily) conjugacy separable groups, some of which we mention in corollaries below.

For any Coxeter group  $W$ , Niblo and Reeves [27] constructed a cube complex  $\mathcal{C}$  on which  $W$  acts properly, and proved that the quotient complex  $\mathcal{X} = W \backslash \mathcal{C}$  is compact if  $W$  is hyperbolic. It follows that any hyperbolic Coxeter group is virtually compact special (originally this is due to Haglund and Wise [15]), hence we can use Theorem 1.1 to deduce:

**Corollary 1.2.** *Any hyperbolic Coxeter group is hereditarily conjugacy separable.*

Note that conjugacy separability of hyperbolic even Coxeter groups was proved in [6].

Another family of hyperbolic virtually compact special groups is given by groups with finite small cancellation presentations. Indeed, in [33] Wise proved that many classical small cancellation groups, including  $C'(1/6)$  and  $C'(1/4) - T(4)$  groups, act properly and cocompactly on CAT(0) cube complexes. It is well-known that such groups are hyperbolic, so Agol’s result [1, Thm. 1.1] applies and, together with Theorem 1.1, it yields

**Corollary 1.3.** *Let  $G$  be a group with a finite  $C'(1/6)$  or  $C'(1/4) - T(4)$  presentation. Then  $G$  is hereditarily conjugacy separable.*

Finally, Theorem 1.1 implies that any group acting properly and cocompactly on the hyperbolic 3-space is hereditarily conjugacy separable, because fundamental groups of closed hyperbolic 3-manifolds are virtually compact special by a combination of results of Bergeron and Wise [3] and Agol [1]. Thus we obtain the following statement:

**Corollary 1.4.** *Any uniform lattice in  $PSL_2(\mathbb{C})$  is hereditarily conjugacy separable.*

The above corollary could also be proved by combining results of Chagas and the second author [7, Thm. 2.5 or Thm. 2.7] with a different theorem of Agol from [1], claiming that closed hyperbolic 3-manifolds are virtually fibered.

Let us now say a few words about the proof of Theorem 1.1. One of the main difficulties in it is to separate conjugacy classes of torsion elements in a finite quotient. To this end we come up with a new approach (see Proposition 3.2) which employs (co)homological methods and is based on a result of K.S. Brown [5] allowing one to distinguish conjugacy classes of elements of prime order using group cohomology. In particular we obtain the following quite general result.

**Theorem 1.5.** *Let  $G$  be a residually finite group with  $\text{vcd}(G) < \infty$ . If  $G$  is cohomologically good then every element of prime order is conjugacy distinguished in  $G$ .*

Recall that a residually finite group  $G$  is *cohomologically good*, if the inclusion of  $G$  in its profinite completion induces an isomorphism on cohomology with finite coefficients. An element  $g \in G$  is said to be *conjugacy distinguished* if the conjugacy class  $g^G$  is closed in the profinite topology on  $G$  (thus  $G$  is conjugacy separable if and only if each  $g \in G$  is conjugacy distinguished). The claim of Theorem 1.5 can be restated by saying that two non-conjugate elements of prime order in  $G$  are not conjugate in the profinite completion  $\widehat{G}$ ; in other words, the embedding of  $G$  in  $\widehat{G}$  induces an injective map on the sets of conjugacy classes of elements of prime order in  $G$  and in  $\widehat{G}$ . In Corollary 3.5 we prove that if, additionally,  $G$  is finitely generated then this map is actually a bijection (in particular, every element of prime order in  $\widehat{G}$  is conjugate to some element in  $G$ ).

To prove Theorem 1.1 for a hyperbolic virtually compact special group  $G$ , we first show that  $G$  is cohomologically good by proving that this property is stable under virtual retractions (Lemma 3.1), and combining this with some results from [16, 14, 20] (our argument actually does not make use of the hyperbolicity of  $G$  and works, more generally, for almost virtual retracts of right angled Artin groups – see Proposition 3.8). It follows that Theorem 1.5 can be applied to separate the conjugacy classes of elements of prime order in  $G$ . After this we prove that every torsion element of  $G$  is conjugacy distinguished essentially by induction on its order.

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## 2. PRELIMINARIES

**2.1. Notation.** Given a group  $G$ , its subgroups  $K, H$  and an element  $g \in G$ , we will write  $C_H(g) = \{h \in H \mid hgh^{-1} = g\}$  to denote the *centralizer* of  $g$  in  $H$ , and  $N_H(K) = \{h \in H \mid hKh^{-1} = K\}$  to denote the *normalizer* of  $K$  in  $H$ .

**2.2. Hyperbolic groups and quasiconvex subgroups.** Recall that a geodesic metric space  $Y$  is (Gromov) *hyperbolic* if there exists a constant  $\delta \geq 0$  such that for any geodesic

triangle  $\Delta$  in  $Y$ , any side of  $\Delta$  is contained in the closed  $\delta$ -neighborhood of the union of the other sides (cf. [2]). A subset  $Z \subseteq Y$  is *quasiconvex* if there is  $\varepsilon \geq 0$  such that for any two points  $z_1, z_2 \in Z$ , any geodesic joining these points is contained in the closed  $\varepsilon$ -neighborhood of  $Z$ .

If  $G$  is a group generated by a finite set  $\mathcal{A} \subseteq G$ , then  $G$  is said to be *hyperbolic* if its Cayley graph  $\Gamma(G, \mathcal{A})$  is a hyperbolic metric space. Similarly, a subset  $S \subseteq G$  is *quasiconvex* if it is quasiconvex when considered as a subset of  $\Gamma(G, \mathcal{A})$ .

Quasiconvex subgroups are very important in the study of hyperbolic groups. Such subgroups are themselves hyperbolic and are quasi-isometrically embedded in  $G$  (see [2]). Basic examples of quasiconvex subgroups in hyperbolic groups are centralizers of elements (see [4, Ch. III.Γ, Prop. 3.9]); this fact will be important for our argument below.

**2.3. Right angled Artin groups.** A right angled Artin group is a group which can be given by a finite presentation, where the only defining relators are commutators of the generators. To construct such a group, one usually starts with a finite simplicial graph  $\Gamma$  with vertex set  $V$  and edge set  $E$ . One then defines the corresponding *right angled Artin group*  $A = A(\Gamma)$  by the following presentation:

$$A = \langle V \mid [u, v] = 1, \text{ whenever } (u, v) \in E \rangle,$$

where  $[u, v] = uvu^{-1}v^{-1}$  is the commutator of  $u$  and  $v$ .

For any subset  $S \subseteq V$ , the subgroup  $A_S = \langle S \rangle \leq A$  is said to be a *full subgroup* of  $A$ . It is easy to see that  $A_S$  is itself a right angled Artin group corresponding to the full subgraph  $\Gamma_S$  of  $\Gamma$ , induced by the vertices from  $S$ . Moreover,  $A_S$  is a retract of  $A$  – see [25, Sec. 6].

Recall that a subgroup  $H$ , of a group  $G$ , is a *virtual retract* if  $H$  is a retract of some finite index subgroup  $K \leq G$ . In other words,  $H \subseteq K$  and there is a homomorphism  $\rho : K \rightarrow H$  such that  $\rho(K) = H$  and  $\rho|_H = \text{id}_H$ .

Let  $\mathcal{VR}$  denote the class of all groups which are virtual retracts of finitely generated right angled Artin groups, and let  $\mathcal{AVR}$  be the class consisting of all groups  $G$  such that  $G$  has a finite index subgroup from  $\mathcal{VR}$ . We are interested in these specific classes of groups because of the following two results: in [16] Haglund and Wise proved that any virtually compact special group  $G$  belongs to the class  $\mathcal{AVR}$ , and in [25] the first author showed that any group  $H \in \mathcal{VR}$  is hereditarily conjugacy separable.

**2.4. Profinite topology.** The *profinite topology* on a group  $G$  is defined by taking finite index subgroups as a basis of neighborhoods of the identity element. This topology is Hausdorff, i.e.,  $\{1\}$  is a closed subset of  $G$ , if and only if the group  $G$  is residually finite. In the latter case,  $G$  embeds in its profinite completion,  $\widehat{G}$ , and the profinite topology on  $G$  is precisely the restriction of the natural topology of  $\widehat{G}$  to  $G$ .

A subset  $S \subseteq G$  is said to be *separable* if it is closed in the profinite topology on  $G$ . Thus an element  $x \in G$  is conjugacy distinguished if its conjugacy class  $x^G = \{gxg^{-1} \mid g \in G\}$  is separable in  $G$ . It is not difficult to see that the latter is equivalent to the property that for any element  $y \in G$ , which is not conjugate to  $x$ , there is a finite group  $F$  and a

homomorphism  $\phi : G \rightarrow F$ , such that  $\phi(y)$  is not conjugate to  $\phi(x)$  in  $F$ . It follows that  $G$  is conjugacy separable if and only if all of its elements are conjugacy distinguished.

**2.5. Criteria for conjugacy separability.** The next standard observation will be useful (cf. [24, Lemma 7.2]):

**Lemma 2.1.** *Let  $K$  be a subgroup of finite index in a group  $G$  and let  $x \in K$ . If  $x$  is conjugacy distinguished in  $K$  then  $x$  is conjugacy distinguished in  $G$ .*

The following criterion was discovered by Chagas and the second author in [7]:

**Proposition 2.2** ([7, Prop. 2.1]). *Let  $H$  be a normal subgroup of index  $m \in \mathbb{N}$  in a group  $G$  and let  $x \in G$  be any element. Suppose that  $H$  is hereditarily conjugacy separable and the centralizer  $C_G(x^m)$ , of  $x^m \in H$ , satisfies the following conditions:*

- (i)  *$x$  is conjugacy distinguished in  $C_G(x^m)$ ;*
- (ii) *each finite index subgroup of  $C_G(x^m)$  is separable in  $G$ .*

*Then  $x$  is conjugacy distinguished in  $G$ .*

Note that the original condition (i) from [7, Prop. 2.1] required  $C_G(x^m)$  to be conjugacy separable, however, it is easy to see that the proof (see also [6, Prop. 2.2] for an alternative argument) only uses the weaker assumption that  $x$  is conjugacy distinguished in  $C_G(x^m)$ .

**2.6. Profinite topology on virtually compact special groups.** Let  $\mathcal{VCSH}$  denote the class of all virtually compact special hyperbolic groups.

*Remark 2.3.* The class  $\mathcal{VCSH}$  is closed under taking finite index subgroups and overgroups.

Indeed, it is immediate from the definitions that a finite index subgroup/overgroup of a virtually compact special group is still virtually compact special. On the other hand, it is well-known that a group is hyperbolic if and only if a finite index subgroup is hyperbolic (for instance, this follows from the fact that hyperbolicity is invariant under quasi-isometries – see [4, Ch. III.H, Thm. 1.9]).

The next statement easily follows from the work of Haglund and Wise in [16].

**Lemma 2.4.** *Suppose that  $G \in \mathcal{VCSH}$  and  $g \in G$ . Then*

- (a) *the centralizer  $C_G(g)$  also belongs to  $\mathcal{VCSH}$ ;*
- (b) *every finite index subgroup of  $C_G(g)$  is separable in  $G$ .*

*Proof.* Fix some finite generating set  $\mathcal{A}$  of  $G$ . Since the group  $G$  is hyperbolic, it is well-known that centralizers of elements in  $G$  are quasiconvex (see, for example, [4, Ch. III.Γ, Prop. 3.9]). Hence  $C_G(g)$  is quasiconvex, so it is also hyperbolic (cf. [2, Lemma 3.8]). In [16, Cor. 7.8] Haglund and Wise proved that any quasiconvex subgroup of  $G$  is virtually compact special, thus (a) is proved.

To prove (b), note that every finite index subgroup  $N \leq C_G(g)$  is also quasiconvex (because there is a constant  $c \geq 0$  such that every element of  $C_G(g)$  is at distance no more

than  $c$  from an element of  $N$  in the Cayley graph  $\Gamma(G, \mathcal{A})$ . Therefore  $N$  is separable in  $G$  by [16, Cor. 7.4 and Lemma 7.5].  $\square$

**Lemma 2.5.** *Any virtually compact special group  $G$  has a finite index normal subgroup  $H \triangleleft G$  such that  $H \in \mathcal{VR}$ ,  $H$  is torsion-free and hereditarily conjugacy separable.*

*Proof.* In [16] Haglund and Wise proved that every virtually compact special group  $G$  has a finite index normal subgroup  $H \triangleleft G$  such that  $H \in \mathcal{VR}$ . Now,  $H$  is torsion-free as right angled Artin groups are torsion-free, and  $H$  is hereditarily conjugacy separable by [25, Cor. 2.1].  $\square$

### 3. COHOMOLOGICAL GOODNESS AND ITS APPLICATIONS TO CONJUGACY SEPARABILITY

Recall that a group  $G$  is cohomologically good, if the natural embedding  $G \hookrightarrow \widehat{G}$ , of the group in its profinite completion, induces an isomorphism on cohomology with finite coefficients. This notion was originally introduced by Serre in [30, Exercises in Sec. I.2.6].

Cohomological goodness of residually finite groups behaves nicely under certain free constructions and is stable under group commensurability (see [14, 20]). We begin this section with proving another useful permanence property:

**Lemma 3.1.** *Suppose that  $G$  is a residually finite cohomologically good group and  $H$  is a virtual retract of  $G$ . Then  $H$  is cohomologically good.*

*Proof.* Since the cohomological goodness passes to subgroups of finite index (see [14, Lemma 3.2]), we may assume that  $H$  is a retract of  $G$ . Let  $f : G \rightarrow H$  be a retraction. Then the profinite topology on  $G$  induces the full profinite topology on  $H$  (see, for example, [28, Lemma 3.1.5]), hence the natural embedding  $i : H \rightarrow G$  induces an injective continuous map  $\widehat{i} : \widehat{H} \rightarrow \widehat{G}$  (cf. [28, Lemma 3.2.6]). Therefore, the functorial property of profinite completions shows that the retraction  $f$  induces a retraction  $\widehat{f} : \widehat{G} \rightarrow \widehat{H}$ , giving rise to the following commutative diagram, where the vertical maps are the natural embeddings of the residually finite groups in their profinite completions:

$$(1) \quad \begin{array}{ccc} \widehat{H} & \xrightleftharpoons[\widehat{f}]{\widehat{i}} & \widehat{G} \\ \uparrow & & \uparrow \\ H & \xrightleftharpoons[f]{i} & G \end{array}$$

If  $M$  is a finite  $H$ -module, we can turn it into a  $G$ -module by letting the kernel of  $f$  act trivially on  $M$ . Then for any  $n \in \mathbb{N} \cup \{0\}$ , (1) induces the following commutative diagram of cohomology groups:



$$\begin{array}{ccc}
 H^n(\widehat{H}, M) & \xrightleftharpoons[\widehat{i}^*]{\widehat{f}^*} & H^n(\widehat{G}, M) \\
 \downarrow \text{res}_{\widehat{H}}^{\widehat{H}} & & \downarrow \text{res}_{\widehat{G}}^{\widehat{G}} \\
 H^n(H, M) & \xrightleftharpoons[i^*]{f^*} & H^n(G, M)
 \end{array}$$

Since  $f \circ i = \text{id}_H$  and  $\widehat{f} \circ \widehat{i} = \text{id}_{\widehat{H}}$ , we can deduce that  $i^* \circ f^*$  and  $\widehat{i}^* \circ \widehat{f}^*$  are identity maps on  $H^n(H, M)$  and  $H^n(\widehat{H}, M)$  respectively. In particular, the map  $\widehat{f}^*$  is injective and the map  $i^*$  is surjective.

Since  $G$  is cohomologically good the right vertical arrow is a bijection and we need to show that so is the left vertical arrow. To see the injectivity, pick an element  $h \in H^n(\widehat{H}, M)$ . Then  $(f^* \circ \text{res}_{\widehat{H}}^{\widehat{H}})(h) = (\text{res}_G^{\widehat{G}} \circ \widehat{f}^*)(h)$ , implying that  $h = 0$  if  $\text{res}_{\widehat{H}}^{\widehat{H}}(h) = 0$ .

For surjectivity, observe that  $i^* \circ \text{res}_G^{\widehat{G}} = \text{res}_H^{\widehat{H}} \circ \widehat{i}^*$  and the map on the left-hand side is surjective, hence  $\text{res}_H^{\widehat{H}}$  must also be surjective.

Thus  $\text{res}_H^{\widehat{H}}$  is an isomorphism, as required.  $\square$

The next statement establishes a connection between cohomological goodness and separability of conjugacy classes of elements of prime order.

**Proposition 3.2.** *Let  $G$  be a residually finite cohomologically good group of finite virtual cohomological dimension. Suppose that  $G$  splits as a semidirect product  $G = H \rtimes \langle x \rangle$ , where  $H \triangleleft G$  is torsion-free and  $x \in G$  has prime order  $p$ . Then the natural embedding of  $G$  in  $\widehat{G}$  induces an injective map between the conjugacy classes of finite subgroups in  $G$  and in  $\widehat{G}$ .*

*Proof.* Fix any integer  $n > \text{vcd}(G)$ . Let  $I$  [respectively,  $\widehat{I}$ ] denote the set of conjugacy classes of subgroups of order  $p$  in  $G$  [respectively, in  $\widehat{G}$ ]. For every conjugacy class  $\alpha \in I$  choose any subgroup  $C_\alpha$ , of order  $p$ , representing it in  $G$ . Since all elementary abelian  $p$ -subgroups of  $G$  have rank at most 1 (as  $G = H \rtimes \langle x \rangle$  and  $H$  is torsion-free), we can apply a classical result of Brown (cf. Cor. 7.4 and the Remark below it in Ch. X of [5]), claiming that there is a canonical isomorphism

$$(2) \quad \eta : H^n(G, \mathbb{Z}/p) \rightarrow \prod_{\alpha \in I} H^n(N_G(C_\alpha), \mathbb{Z}/p).$$

Denote  $N_\alpha = N_G(C_\alpha)$ ,  $\alpha \in I$ . The above isomorphism  $\eta$  can be defined as follows: for each  $\alpha \in I$ , the inclusion  $N_\alpha \hookrightarrow G$  induces the restriction homomorphism  $\text{res}_{N_\alpha}^G : H^n(G, \mathbb{Z}/p) \rightarrow H^n(N_\alpha, \mathbb{Z}/p)$ , and  $\eta = \prod_{\alpha \in I} \text{res}_{N_\alpha}^G$  is the corresponding diagonal map.

For our purposes, it is actually more convenient to work with homology instead of cohomology. For each  $\alpha \in I$ , the inclusion  $N_\alpha \hookrightarrow G$  induces the corestriction homomorphism

$\text{cor}_{N_\alpha}^G : H_n(N_\alpha, \mathbb{Z}/p) \rightarrow H_n(G, \mathbb{Z}/p)$ . This gives a natural homomorphism

$$(3) \quad \varphi : \bigoplus_{\alpha \in I} H_n(N_\alpha, \mathbb{Z}/p) \rightarrow H_n(G, \mathbb{Z}/p),$$

defined by the property that the restriction of  $\varphi$  to each direct summand  $H_n(N_\alpha, \mathbb{Z}/p)$  is the map  $\text{cor}_{N_\alpha}^G$ .

Since  $\mathbb{Z}/p$  is a field, the contravariant functor  $\text{Hom}_{\mathbb{Z}/p}(-, \mathbb{Z}/p)$  induces a natural isomorphism between  $\text{Hom}_{\mathbb{Z}/p}(H_n(F, \mathbb{Z}/p), \mathbb{Z}/p)$  and  $H^n(F, \mathbb{Z}/p)$  for any group  $F$  (for example by the Universal Coefficient Theorem, cf. [18, Sec. 3.1, pp. 196-197]). Applying this functor to (3) gives the map  $\eta$  from (2).

If the map  $\varphi$  was not injective then we would have a short exact sequence

$$\{0\} \rightarrow K \rightarrow \bigoplus_{\alpha \in I} H_n(N_\alpha, \mathbb{Z}/p) \xrightarrow{\varphi} H_n(G, \mathbb{Z}/p) \rightarrow \{0\},$$

where  $K$  is a non-trivial vector space over  $\mathbb{Z}/p$ . Since  $\mathbb{Z}/p$  is a field, the functor  $\text{Hom}_{\mathbb{Z}/p}(-, \mathbb{Z}/p)$  is exact, so it would give a short exact sequence

$$\{0\} \rightarrow H^n(G, \mathbb{Z}/p) \xrightarrow{\eta} \prod_{\alpha \in I} H^n(N_\alpha, \mathbb{Z}/p) \rightarrow \text{Hom}_{\mathbb{Z}/p}(K, \mathbb{Z}/p) \rightarrow \{0\}.$$

The latter would contradict the fact that  $\eta$  is surjective, as  $\text{Hom}_{\mathbb{Z}/p}(K, \mathbb{Z}/p) \neq \{0\}$ . Therefore  $\varphi$  is injective. A similar argument shows that  $\varphi$  is also surjective, as  $\eta$  is injective. Hence the homomorphism  $\varphi$  in (3) is an isomorphism.

In particular, we see that if  $\alpha_1$  and  $\alpha_2$  are distinct elements of  $I$  then

$$(4) \quad \varphi(H_n(N_{\alpha_1}, \mathbb{Z}/p)) \cap \varphi(H_n(N_{\alpha_2}, \mathbb{Z}/p)) = \{0\} \text{ in } H_n(G, \mathbb{Z}/p).$$

By the assumptions, for each  $k = 1, 2$ ,  $G = H \rtimes C_{\alpha_k}$ , i.e.,  $G$  retracts onto  $C_{\alpha_k}$ . Therefore  $N_{\alpha_k}$  also retracts onto  $C_{\alpha_k}$ , and hence the corestriction homomorphism  $\text{cor}_{C_{\alpha_k}}^{N_{\alpha_k}} : H_n(C_{\alpha_k}, \mathbb{Z}/p) \rightarrow H_n(N_{\alpha_k}, \mathbb{Z}/p)$  is injective. Since  $H_n(C_{\alpha_k}, \mathbb{Z}/p) \neq \{0\}$  for  $k = 1, 2$  (as  $C_{\alpha_k} \cong \mathbb{Z}/p$ ), (4) shows that the natural images of  $H_n(C_{\alpha_1}, \mathbb{Z}/p)$  and  $H_n(C_{\alpha_2}, \mathbb{Z}/p)$  in  $H_n(G, \mathbb{Z}/p)$  must be distinct.

Now, arguing by contradiction, assume that there exist distinct  $\alpha_1, \alpha_2 \in I$  such that  $C_{\alpha_1}$  is conjugate to  $C_{\alpha_2}$  in  $\widehat{G}$ . We have the following commutative diagram coming from the natural inclusions:

$$(5) \quad \begin{array}{ccc} & \widehat{G} & \\ & \uparrow & \\ C_{\alpha_1} & \xrightarrow{\quad} G \xleftarrow{\quad} & C_{\alpha_2} \end{array} .$$

Since  $C_{\alpha_k}$  is a closed subgroup of  $\widehat{G}$ ,  $k = 1, 2$ , and  $G$  is dense in  $\widehat{G}$ , this diagram induces the following commutative diagram of cohomology groups (for the vertical and diagonal arrows see [30, Sec. I.2.4 and Exercise 1] in Sec. I.2.6):



$$(6) \quad \begin{array}{ccccc} & & H^n(\widehat{G}, \mathbb{Z}/p) & & \\ & \swarrow \text{res}_{C_{\alpha_1}}^{\widehat{G}} & \downarrow \text{res}_{\widehat{G}}^{\widehat{G}} & \searrow \text{res}_{C_{\alpha_2}}^{\widehat{G}} & \\ H^n(C_{\alpha_1}, \mathbb{Z}/p) & \xleftarrow{\text{res}_{C_{\alpha_1}}^G} & H^n(G, \mathbb{Z}/p) & \xrightarrow{\text{res}_{C_{\alpha_k}}^G} & H^n(C_{\alpha_2}, \mathbb{Z}/p) \end{array},$$

where  $\text{res}_{\widehat{G}}^{\widehat{G}}$  is an isomorphism by cohomological goodness of  $G$ .

Let us apply the  $\text{Hom}_{\mathbb{Z}/p}(-, \mathbb{Z}/p)$  functor to the diagram (6). Pontryagin duality between cohomology and homology of profinite groups (see [28, Prop. 6.3.6]) says that  $\text{Hom}_{\mathbb{Z}/p}(H^n(\widehat{G}, \mathbb{Z}/p), \mathbb{Z}/p)$  is naturally isomorphic to  $H_n(\widehat{G}, \mathbb{Z}/p)$ . On the other hand, for the discrete group  $G$ ,  $\text{Hom}_{\mathbb{Z}/p}(H^n(G, \mathbb{Z}/p), \mathbb{Z}/p)$  may not be, in general, isomorphic to  $H_n(G, \mathbb{Z}/p)$ . However, since  $\text{Hom}_{\mathbb{Z}/p}(H_n(G, \mathbb{Z}/p), \mathbb{Z}/p) \cong H^n(G, \mathbb{Z}/p)$  (as observed above), the space  $\text{Hom}_{\mathbb{Z}/p}(H^n(G, \mathbb{Z}/p), \mathbb{Z}/p)$  can be thought of as the double dual of  $H_n(G, \mathbb{Z}/p)$ . Since there is always a canonical embedding of a vector space into its double dual, we obtain an injective homomorphism  $\rho : H_n(G, \mathbb{Z}/p) \rightarrow H_n(\widehat{G}, \mathbb{Z}/p)$ , which fits into the following commutative diagram:

$$(7) \quad \begin{array}{ccccc} & & H_n(\widehat{G}, \mathbb{Z}/p) & & \\ & \nearrow \hat{\tau}_1 & \uparrow \rho & \nwarrow \hat{\tau}_2 & \\ H_n(C_{\alpha_1}, \mathbb{Z}/p) & \xrightarrow{\tau_1} & H_n(G, \mathbb{Z}/p) & \xleftarrow{\tau_2} & H_n(C_{\alpha_2}, \mathbb{Z}/p) \end{array},$$

where  $H_n(\widehat{G}, \mathbb{Z}/p)$  is the profinite homology of  $\widehat{G}$ ,  $\tau_k = \text{cor}_{C_{\alpha_k}}^G$  and  $\hat{\tau}_k = \text{cor}_{C_{\alpha_k}}^{\widehat{G}}$ ,  $k = 1, 2$ .

By the assumption, there exists  $g \in \widehat{G}$  such that  $C_{\alpha_2} = gC_{\alpha_1}g^{-1}$ . Hence we have

$$\begin{array}{ccc} \widehat{G} & \longleftrightarrow & C_{\alpha_1} \\ \downarrow i_g & & \downarrow i_g|_{C_{\alpha_1}} \\ \widehat{G} & \longleftrightarrow & C_{\alpha_2} \end{array},$$

where  $i_g : \widehat{G} \rightarrow \widehat{G}$  is the inner automorphism of  $\widehat{G}$  given by  $i_g(h) = ghg^{-1}$ , for all  $h \in \widehat{G}$ , and  $i_g|_{C_{\alpha_1}} : C_{\alpha_1} \rightarrow C_{\alpha_2}$  is its restriction to  $C_{\alpha_1}$ . This leads to the following commutative diagram between the corresponding homology groups:

$$\begin{array}{ccc} H_n(\widehat{G}, \mathbb{Z}/p) & \xleftarrow{\hat{\tau}_1} & H_n(C_{\alpha_1}, \mathbb{Z}/p) \\ \downarrow \text{id} & & \downarrow \cong \\ H_n(\widehat{G}, \mathbb{Z}/p) & \xleftarrow{\hat{\tau}_2} & H_n(C_{\alpha_2}, \mathbb{Z}/p) \end{array}$$

Note that the left vertical map is the identity on  $H_n(\widehat{G}, \mathbb{Z}/p)$ , as it is induced by an inner automorphism of  $\widehat{G}$  (this is easy to prove directly, or one can use [30, Exercise 1] in Sec. I.2.5] and apply the Pontryagin duality between  $H^n$  and  $H_n$ ). Therefore we can conclude that  $\hat{\tau}_1(H_n(C_{\alpha_1}, \mathbb{Z}/p)) = \hat{\tau}_2(H_n(C_{\alpha_2}, \mathbb{Z}/p))$  in  $H_n(\widehat{G}, \mathbb{Z}/p)$ . Thus, in view of

injectivity of the map  $\rho$  from (7), in  $H_n(G, \mathbb{Z}/p)$  we must have that  $\tau_1(H_n(C_{\alpha_1}, \mathbb{Z}/p)) = \tau_2(H_n(C_{\alpha_2}, \mathbb{Z}/p))$ . The latter gives a contradiction with the property that the natural images of  $H_n(C_{\alpha_1}, \mathbb{Z}/p)$  and  $H_n(C_{\alpha_2}, \mathbb{Z}/p)$  in  $H_n(G, \mathbb{Z}/p)$  are distinct, which was proved above as a consequence of the fact that the map  $\varphi$  in (3) is injective.

Therefore,  $C_{\alpha_1}$  cannot be conjugate to  $C_{\alpha_2}$  in  $\widehat{G}$  if  $\alpha_1 \neq \alpha_2$  in  $I$ . This means that the inclusion  $G \hookrightarrow \widehat{G}$  induces an injective map from  $I$  to  $\widehat{I}$ , as required.  $\square$

We are now ready to prove Theorem 1.5, stated in the Introduction.

*Proof of Theorem 1.5.* Let  $p$  be a prime and let  $x$  be an element of order  $p$  in  $G$ . By the assumptions there exists a torsion-free normal subgroup  $H \triangleleft G$ , which has finite index in  $G$ . Denote  $G_1 = H\langle x \rangle \leq G$ . Clearly  $G_1$  has finite index in  $G$ , and  $G_1 \cong H \rtimes \langle x \rangle$ . Therefore  $G_1$  is residually finite and  $\text{vcd}(G_1) = \text{vcd}(G) < \infty$ . Moreover,  $G_1$  is cohomologically good since this property passes to finite index subgroups and overgroups (see [14, Lemma 3.2]). Thus the group  $G_1$  satisfies all the assumptions of Proposition 3.2.

Consider any element  $y \in G_1$ , which is not conjugate to  $x$ . If  $y$  and  $x$  have different orders, then, using residual finiteness of  $G_1$ , we can find a finite quotient  $M$ , of  $G_1$ , where the images of  $y$  and  $x$  still have different orders, and hence they will not be conjugate in  $M$ . Therefore in this case  $M$  will be a finite quotient of  $G_1$  distinguishing the conjugacy classes of  $y$  and  $x$ .

So, now we can suppose that  $y$  also has order  $p$ . If  $\langle y \rangle$  is not conjugate to  $\langle x \rangle$  in  $G_1$ , then, by Proposition 3.2, these subgroups are also not conjugate in  $\widehat{G}_1$ . Hence  $y$  is not conjugate to  $x$  in  $\widehat{G}_1$ , i.e.,  $y \notin x^{\widehat{G}_1}$ . Now, the conjugacy class  $x^{\widehat{G}_1}$  is closed in  $\widehat{G}_1$ , as  $\widehat{G}_1$  is compact, so  $x^{\widehat{G}_1} \cap G_1$  is a separable subset of  $G_1$  which contains  $x^{G_1}$  but avoids  $y$ . It follows that there is a finite quotient of  $G_1$  distinguishing the conjugacy classes of  $x$  and  $y$ .

Thus we can further assume that  $\langle y \rangle$  is conjugate to  $\langle x \rangle$  in  $G_1$ . Then  $hyh^{-1} = z$  for some  $h \in G_1$  and some  $z \in \langle x \rangle$ . Note that  $z \neq x$  as  $y$  is not conjugate to  $x$  in  $G_1$ , by our assumption. Consequently,  $z = \xi(z) \neq \xi(x) = x$ , where  $\xi : G_1 \rightarrow \langle x \rangle$  is the natural retraction (coming from the semidirect product decomposition of  $G_1$ ). Since the group  $\langle x \rangle$  is abelian, we can conclude that  $\xi(y) = \xi(z)$  is not conjugate to  $\xi(x)$  in it, so  $\langle x \rangle$  is a finite quotient of  $G_1$  distinguishing the conjugacy classes of  $x$  and  $y$ .

Thus we have considered all possibilities, showing that  $x$  is conjugacy distinguished in  $G_1$ . It remains to apply Lemma 2.1 to conclude that  $x$  is conjugacy distinguished in  $G$ , as required.  $\square$

Proposition 3.2 shows that, under its assumptions, the natural inclusion  $G \rightarrow \widehat{G}$  induces an injective map between the conjugacy classes of prime order subgroups in  $G$  and in  $\widehat{G}$ . To complement this, we will now show this map is also surjective, provided  $G$  has finitely many conjugacy classes of elements of prime order (the latter will be satisfied if  $G$  is finitely generated – see Corollary 3.5 below).

**Lemma 3.3.** *Suppose that  $H$  is a cohomologically good group with  $\text{cd}(H) = n < \infty$ . Then  $\text{cd}(\widehat{H}) \leq n$ ; in particular,  $\widehat{H}$  is torsion-free.*

*Proof.* If  $A$  is any simple discrete  $\widehat{H}$ -module, then  $A$  is finite (because  $\widehat{H}$  is compact and its action on  $A$  is continuous), so  $H^{n+1}(\widehat{H}, A) \cong H^{n+1}(H, A) = \{0\}$  by cohomological goodness of  $H$  and the assumption that  $\text{cd}(H) < n + 1$ . Hence  $\text{cd}_p(\widehat{H}) \leq n$  for every prime  $p$  by [28, Prop. 7.1.4], therefore

$$\text{cd}(\widehat{H}) := \sup\{\text{cd}_p(\widehat{H}) \mid p \text{ prime}\} \leq n.$$

Finally, since  $\text{cd}_p(C) \leq \text{cd}_p(\widehat{H}) < \infty$  for each prime  $p$  and every closed subgroup  $C \leq \widehat{H}$  (cf. [28, Thm. 7.3.1]), and  $\text{cd}_p(\mathbb{Z}/p) = \infty$  we can conclude that  $\widehat{H}$  cannot contain subgroups of order  $p$ , for any prime  $p$ . Thus  $\widehat{H}$  must be torsion-free, as claimed.  $\square$

**Proposition 3.4.** *Let  $p$  be a prime and let  $G$  be a residually finite cohomologically good group such that  $\text{vcd}(G) < \infty$  and  $G$  contains finitely many conjugacy classes of subgroups (or, equivalently, elements) of order  $p$ . Then every element of order  $p$  in the profinite completion  $\widehat{G}$  is conjugate to some element of  $G$ .*

*Proof.* Arguing by contradiction suppose that there is some element  $\gamma \in \widehat{G}$ , of order  $p$ , such that  $C = \langle \gamma \rangle$  is not conjugate to any subgroup of  $G$ . By the assumptions, only finitely many conjugacy classes  $\mathcal{C}_1, \dots, \mathcal{C}_k$ , of subgroups of order  $p$  in  $\widehat{G}$ , intersect  $G$  non-trivially. Since each  $\mathcal{C}_i$ ,  $i = 1, \dots, k$ , is a compact subset of  $\widehat{G}$ , avoiding the finite subgroup  $C$ , there is a normal open subgroup  $U$  of  $\widehat{G}$  such that  $CU \cap \mathcal{C}_i = \emptyset$  for every  $i = 1, \dots, k$ . Since  $\text{vcd}(G) < \infty$ ,  $G$  contains a normal torsion-free subgroup  $K$  of finite index. Then the closure  $\overline{K}$ , of  $K$  in  $\widehat{G}$ , is naturally isomorphic to  $\widehat{K}$ , and hence it is torsion-free by Lemma 3.3 ( $K$  is cohomologically good by [28, Lemma 3.2.6] and  $\text{cd}(K) = \text{vcd}(G) < \infty$ ). So, after replacing  $U$  by  $U \cap \overline{K}$ , we can assume that  $U$  is torsion-free.

Now,  $CU$  is an open subgroup of  $\widehat{G}$ , so  $H = G \cap CU$  is a finite index subgroup of  $G$ , whose closure  $\overline{H}$  in  $\widehat{G}$  coincides with  $CU$  (see [28, Prop. 3.2.2]). Since  $H \cap \mathcal{C}_i = \emptyset$ ,  $i = 1, \dots, k$ , and every subgroup of order  $p$  in  $G$  is contained in some  $\mathcal{C}_i$ , we can conclude that  $H$  has no elements of order  $p$ . On the other hand, since  $CU$  is an extension of a torsion-free group  $U$  by the cyclic group  $C$ , of order  $p$ , we see that  $CU$  cannot contain non-trivial elements of finite orders other than  $p$ . Recalling that  $H \leq CU$ , allows us to conclude that  $H$  is torsion-free.

Since  $|G : H| < \infty$  we can argue as in the case of  $K$  above (using Lemma 3.3) to deduce that  $\overline{H} = CU$  must be torsion-free. The latter contradicts the fact that it contains  $C$ , completing the proof of the proposition.  $\square$

**Corollary 3.5.** *Suppose that  $G$  is a finitely generated residually finite cohomologically good group with  $\text{vcd}(G) < \infty$ . Then  $G$  has finitely many conjugacy classes of subgroups of prime power order, and the natural inclusion of  $G$  in  $\widehat{G}$  induces a bijection between the conjugacy classes of elements (or subgroups) of prime order in  $G$  and in  $\widehat{G}$ .*

*Proof.* By the assumptions,  $G$  has a normal torsion-free finite index subgroup  $H$ . It follows that there can be only finitely many primes  $p$  such that  $G$  contains some non-trivial  $p$ -subgroup. Let  $p$  be such a prime. Since  $G$  is cohomologically good, the same is true for  $H$ , so we can use a theorem of Weigel and the second author [32, Thm. B] claiming

that  $H^n(H, \mathbb{Z}/p)$  is finite for every  $n \geq 0$ . Since  $\mathbb{Z}/p$  is a field, the Universal Coefficient Theorem tells us that the  $\mathbb{Z}/p$ -vector space  $H^n(H, \mathbb{Z}/p)$  is the dual of  $H_n(H, \mathbb{Z}/p)$ , hence the latter is also finite. Therefore we can apply a result of Brown [5, Lemma IX.13.2] claiming that  $G$  contains finitely many conjugacy classes of  $p$ -subgroups.

Thus we can use Proposition 3.4, to conclude that the natural map between the conjugacy classes of elements of prime order in  $G$  and in  $\widehat{G}$  is surjective. This map is injective by Theorem 1.5, so the corollary is proved.  $\square$

*Remark 3.6.* In the case when the group  $G$  is virtually of type FP, Thm. 8.2 in the survey paper [19] asserts (without proof) that, with some extra work, a stronger version of Corollary 3.5 can be derived from a general result of Symonds [31, Thm. 1.1] (this was also confirmed to us by Symonds in a private communication).

An important tool for establishing cohomological goodness was discovered by Grunewald, Jaikin-Zapirain and the second author, and, independently, by Lorensen:

**Proposition 3.7** ([14, Prop. 3.6], [20, Cor. 3.11]). *Let  $G = H *_B A^t$  be an HNN-extension of a cohomologically good group  $H$ , where the associated subgroups  $A$  and  $B$  are also cohomologically good. Suppose that  $G$  is residually finite,  $H$ ,  $A$  and  $B$  are separable in  $G$  and the profinite topology on  $G$  induces the full profinite topologies on  $H$ ,  $A$ , and  $B$ . Then  $G$  is cohomologically good.*

This allows us to show that in fact any group from the class  $\mathcal{AVR}$  is cohomologically good.

**Proposition 3.8.** *Let  $G \in \mathcal{AVR}$ . Then  $G$  is residually finite, cohomologically good and has finite virtual cohomological dimension.*

*Proof.* By definition of the class  $\mathcal{AVR}$ , some finite index subgroup  $H \leq G$  is a virtual retract of some right angled Artin group  $A$ . Right angled Artin groups are residually finite (see, for example, [9, Ch. 3, Thm 1.1]), hence  $H$  and  $G$  are both residually finite. The cohomological dimension  $\text{cd}(A)$ , of  $A$ , is equal to the clique number of the associated graph (this follows from the fact that  $A$  acts freely and cocompactly on a  $\text{CAT}(0)$  cube complex of the appropriate dimension – see [8, Sec. 3.6]), therefore  $\text{cd}(H) \leq \text{cd}(A) < \infty$ . Thus  $\text{vcd}(G) = \text{cd}(H) < \infty$ .

To show that  $G$  is cohomologically good, we will first prove this for all right angled Artin groups (cf. [20, Thm. 3.15] and [21]). Let  $B$  be a right angled Artin group corresponding to some finite simplicial graph  $\Gamma$  with vertex set  $V$ . We will show that  $B$  is cohomologically good by induction on  $|V|$ . If  $|V| = 0$  then  $B = \{1\}$  and the claim holds trivially. Now, suppose that  $|V| > 0$  and choose any  $S \subset V$  with  $|V \setminus S| = 1$ . Then  $B$  splits as an HNN-extension of  $B_S$  over another full subgroup  $B_T$ , for some  $T \subset S$  (see [25, Sec. 7]). Since  $B_S$  and  $B_T$  are right angled Artin groups with less than  $|V|$  generators, they are cohomologically good by the induction hypothesis. Recall that both  $B_T$  and  $B_S$  are retracts of  $B$  and  $B$  is residually finite, therefore these subgroups are separable in  $B$  and the profinite topology of  $B$  induces the full profinite topologies on these subgroups (cf. [28, Lemma 3.1.5]). Hence  $B$  is cohomologically good by Proposition 3.7.

Thus we have shown that any right angled Artin group is cohomologically good. Therefore, according to Lemma 3.1, the finite index subgroup  $H \leq G$  is cohomologically good, as a virtual retract of  $A$ . Hence  $G$  is itself cohomologically good by [14, Lemma 3.2].  $\square$

Combining Theorem 1.5 with Proposition 3.8 and Lemma 2.5 we immediately obtain the following statement:

**Corollary 3.9.** *Let  $G$  be a virtually compact special group (or, more generally, let  $G \in \text{AVR}$ ). Then every element of prime order is conjugacy distinguished in  $G$ .*

#### 4. PROOF OF THE MAIN RESULT

Before proving the main result we will need two more auxiliary statements.

**Lemma 4.1.** *Let  $G \in \mathcal{VCSH}$  and let  $x \in G$  be an element of infinite order. Then  $x$  is conjugacy distinguished in  $G$ .*

*Proof.* By Lemma 2.5,  $G$  has a normal subgroup  $H$ , of some finite index  $m \in \mathbb{N}$ , such that  $H$  is hereditarily conjugacy separable. By the assumptions,  $x^m \in H$  is an infinite order element in the hyperbolic group  $G$ , so its centralizer  $C_G(x^m)$  is virtually cyclic (cf. [2, Prop. 3.5]). It follows that  $C_G(x^m)$  is conjugacy separable. The second condition of Proposition 2.2 follows from Lemma 2.4.(b). Therefore we can use this proposition to conclude that  $x$  is conjugacy distinguished in  $G$ , as required.  $\square$

**Corollary 4.2** (cf. [25, Cor. 9.11]). *If  $G \in \mathcal{VCSH}$  and  $H \leq G$  is a torsion-free subgroup of finite index, then  $H$  is hereditarily conjugacy separable.*

*Proof.* Note that  $H \in \mathcal{VCSH}$  by Remark 2.3, hence any element of infinite order is conjugacy distinguished in  $H$  by Lemma 4.1. Since  $H$  is torsion-free, the only element of finite order in  $H$ , the identity element, must also be conjugacy distinguished. Thus all elements of  $H$  are conjugacy distinguished, i.e.,  $H$  is conjugacy separable.

Clearly the same argument applies to any finite index subgroup  $K \leq H$ . Therefore,  $H$  is hereditarily conjugacy separable.  $\square$

*Proof of Theorem 1.1.* Consider any group  $G \in \mathcal{VCSH}$ . Choose a torsion-free normal subgroup  $H \triangleleft G$  such that  $n = |G : H|$  is minimal (such  $H$  exists by Lemma 2.5). We will prove the theorem by induction on  $n$ . If  $n = 1$  the statement holds because  $H$  is hereditarily conjugacy separable by Corollary 4.2. So we can assume that  $n > 1$  and we have already established hereditary conjugacy separability for every group from  $\mathcal{VCSH}$  which has a torsion-free normal subgroup of index less than  $n$ .

We will first show that  $G$  is conjugacy separable. So, consider any element  $x \in G$ . If  $x$  has infinite order, then  $x$  is conjugacy distinguished in  $G$  by Lemma 4.1. Thus we can suppose that  $x$  has finite order.

Set  $K = H\langle x \rangle$  and observe that  $K \in \mathcal{VCSH}$  by Remark 2.3. If  $|K : H| < n$  then  $K$  is hereditarily conjugacy separable by the induction hypothesis, so  $x$  is conjugacy distinguished in  $K$ . But then Lemma 2.1 implies that  $x$  is conjugacy distinguished in  $G$ , as  $|G : K| \leq |G : H| < \infty$ .

Therefore we can assume that  $|K : H| = n = |G : H|$ . It follows that  $G = K$ , i.e.,  $G = H\langle x \rangle \cong H \rtimes \langle x \rangle$ , as  $H$  is torsion-free and  $x$  has finite order (which must then be equal to  $n$ ). We will now consider two cases.

*Case 1:*  $n = p$  is a prime number. Then  $x$  is conjugacy distinguished in  $G$  by Corollary 3.9.

*Case 2:*  $n$  is a composite number. Thus  $n = lm$  for some  $l, m \in \mathbb{N}$ ,  $1 < l, m < n$ . We aim to use the criterion from Proposition 2.2, so let's check that all of its assumptions are satisfied.

Let  $F = H\langle x^m \rangle \leq G$ . Then  $F \in \mathcal{VCSH}$  by Remark 2.3 and  $F \cong H \rtimes (\mathbb{Z}/l)$ . Thus  $F$  is hereditarily conjugacy separable by the induction hypothesis, as  $|F : H| = l < n$ . Evidently,  $F \triangleleft G$  and  $|G : F| = m$ . Every finite index subgroup of  $C_G(x^m)$  is separable in  $G$  by Lemma 2.4.(b), so it remains to check that  $x$  is conjugacy distinguished in  $C_G(x^m)$ .

Set  $H_1 = C_G(x^m) \cap H$ , and observe that  $C_G(x^m) = H_1\langle x \rangle \cong H_1 \rtimes (\mathbb{Z}/n)$ . Moreover, in view of Remark 2.3,  $H_1 \in \mathcal{VCSH}$  as  $|C_G(x^m) : H_1| = n < \infty$  and  $C_G(x^m) \in \mathcal{VCSH}$  by Lemma 2.4.(a).

To verify that  $x$  is conjugacy distinguished in  $C_G(x^m)$ , consider any element  $y \in C_G(x^m)$  which is not conjugate to  $x$  in  $C_G(x^m)$ . Since  $x^m$  is central in  $C_G(x^m)$ , we can let  $L$  be the quotient of  $C_G(x^m)$  by  $\langle x^m \rangle$ , and let  $\phi : C_G(x^m) \rightarrow L$  denote the natural epimorphism.

Clearly  $\phi(H_1) \cong H_1$ , as  $H_1 \cap \ker \phi = \{1\}$ . Therefore  $\phi(H_1)$  is torsion-free and  $L = \phi(H_1)\langle \phi(x) \rangle \cong H_1 \rtimes (\mathbb{Z}/m)$ , implying that  $L \in \mathcal{VCSH}$  (by Remark 2.3). Consequently,  $L$  is hereditarily conjugacy separable by the induction hypothesis, as  $|L : H_1| = m < n$ . Let us again consider two separate subcases.

*Subcase 2.1:* suppose that  $\phi(x)$  and  $\phi(y)$  are not conjugate in  $L$ . Then there is a finite group  $M$  and a homomorphism  $\psi : L \rightarrow M$  such that  $\psi(\phi(x))$  is not conjugate to  $\psi(\phi(y))$  in  $M$ . Thus the homomorphism  $\eta = \psi \circ \phi : C_G(x^m) \rightarrow M$  will distinguish the conjugacy classes of  $x$  and  $y$ , as required.

*Subcase 2.2:* assume that  $\phi(x)$  is conjugate to  $\phi(y)$  in  $L$ . Since  $\ker \phi \subseteq \langle x \rangle$ , we can deduce that there is  $h \in C_G(x^m)$  such that  $hyh^{-1} = z$ , for some  $z \in \langle x \rangle$ .

Now,  $z \neq x$ , since we assumed that  $y$  is not conjugate to  $x$  in  $C_G(x^m)$ . Therefore  $x = \xi(x) \neq \xi(z) = z$ , where  $\xi : C_G(x^m) \rightarrow \langle x \rangle$  is the natural retraction (coming from the decomposition of  $C_G(x^m)$  as a semidirect product of  $H_1$  and  $\langle x \rangle$ ). Recalling that  $\langle x \rangle$  is abelian, we see that  $\xi(y) = \xi(hyh^{-1}) = \xi(z)$ . Therefore  $\xi(y)$  is not conjugate to  $\xi(x)$  in the finite cyclic group  $\langle x \rangle$ . Thus we have distinguished the conjugacy classes of  $x$  and  $y$  in this finite quotient of  $C_G(x^m)$ .

Subcases 2.1 and 2.2 together imply that  $x$  is conjugacy distinguished in  $C_G(x^m)$ . Therefore we have verified all of the assumptions of Proposition 2.2 (for  $G$  and the finite index normal subgroup  $F \triangleleft G$ ), so we can apply this proposition to deduce that  $x$  is conjugacy distinguished in  $G$ . Thus Case 2 is completed.

Cases 1 and 2 exhaust all possibilities, so we have established conjugacy separability for any group  $G \in \mathcal{VCSH}$ , which possesses a torsion-free normal subgroup  $H \triangleleft G$  of index  $n$ . If  $K \leq G$  is any subgroup of finite index, then  $K \in \mathcal{VCSH}$  by Remark 2.3 and  $H \cap K$  is a torsion-free normal subgroup in  $K$  of index at most  $n$ . So, either using the induction



hypothesis (if  $|K : (H \cap K)| < n$ ) or the above argument (if  $|K : (H \cap K)| = n$ ), we can conclude that  $K$  is conjugacy separable as well. Hence  $G$  is hereditarily conjugacy separable, and the step of induction has been established. This finishes the proof of the theorem.  $\square$

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